MAGNETIC FORCE ON TWO STAGGERED SLOTTED HALF-PLANES

by

G. W. Veltkamp* and J. J. A. M. Brands*

1. The problem

We consider the two-dimensional magnetic induction $B = -\text{grad}\,\psi$ in the domain $\mathcal{D}$ between four permeable quarter planes arranged as shown in Fig. 1, each of which is kept at a given constant potential. The two upper quarter planes are rigidly connected and so are the two lower quarter planes.

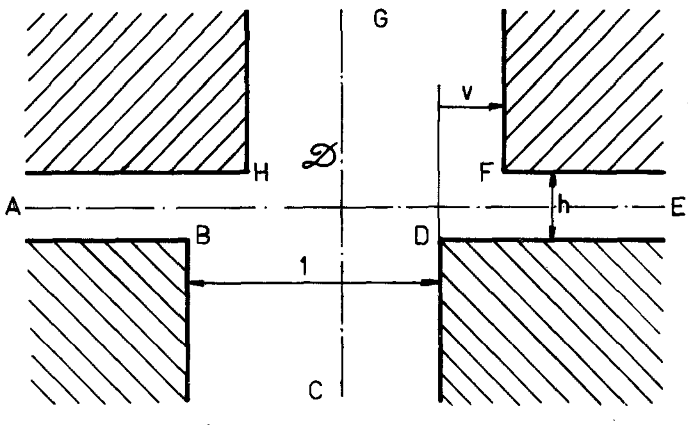

Fig. 1.

Our aim is to find the resulting horizontal force exerted by the field on the two lower quarter planes. In the case of electrical machines with various types of exitation the centring force can be approximated from these results. [1)]

2. A formula for the magnetic force

The magnetic force $\underline{K} = (K_x, K_y)$ exerted on a part AB of the boundary of $\mathcal{D}$ (cf. Fig. 2) along which ψ = constant, is given by

$$\underline{K} = \tfrac{1}{2} \int_A^B |\nabla\psi|^2 \, \underline{n} \, ds \,,$$

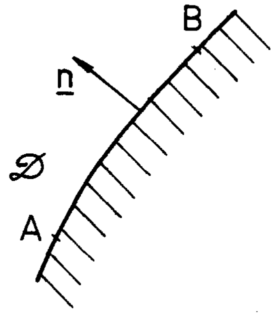

Fig. 2.

1) This problem was suggested by A.J.C. Bakhuizen and P.A.F.M. Goemans, Department of Electrical Engineering, Technological University, Eindhoven.

*) Technological University, Eindhoven, the Netherlands.

where $\underline{n}=(n_x, n_y)$ is the normal on the boundary directed towards the field. From this formula we shall derive another which is more suitable for two-dimensional problems.[1)]

Let $z=x+iy$. Let $\Omega=\Omega(z)$ be an analytic function in the domain $\mathfrak{D}$ such that $\operatorname{Im}\ \Omega(z)=\psi(x,y)$. Let $K=K_x+iK_y$ and $n=n_x+in_y$. Then we have along the boundary $n\,ds=i\,dz$, if dz is taken in the counterclockwise sense relative to $\mathfrak{D}$.

Consequently, since

$$|\nabla\psi|^2=\left|\frac{d\Omega}{dz}\right|^2,$$

we have

$$K=\tfrac{1}{2}i\int_A^B\left|\frac{d\Omega}{dz}\right|^2 dz=\tfrac{1}{2}i\int_A^B\frac{d\overline{\Omega}}{d\overline{z}}\,d\Omega.$$

Since ψ is constant along AB we here have $d\overline{\Omega}=d\Omega$, hence

$$\overline{K}=K_x-iK_y=-\tfrac{1}{2}i\int_A^B\frac{d\Omega}{dz}\,d\Omega=-\tfrac{1}{2}i\int_A^B\left(\frac{d\Omega}{dz}\right)^2 dz\,.$$

Now, it should be remarked that the integrand is analytic in $\mathfrak{D}$, hence for the path of integration we may take any curve in $\mathfrak{D}$ that connects A and B. In the situation sketched in Fig. 1 we want to find the horizontal force on the lower slot BCD. This is defined as

$$K=\operatorname{Re}\left[-\tfrac{1}{2}i\lim_{\substack{C_1\to C\\ C_2\to C}}\left(\int_B^{C_1}+\int_{C_2}^{D}\right)\left(\frac{d\Omega}{dz}\right)^2 dz\right],$$

where C_1 is on BC and C_2 on CD in such a way that C_1C_2 is horizontal.

Since in all considered cases $\lim\limits_{z\to C}\dfrac{d\Omega}{dz}$ exists and is purely imaginary we have

$$\operatorname{Re}\left[-\tfrac{1}{2}i\lim_{\substack{C_1\to C\\ C_2\to C}}\int_{C_1}^{C_2}\right]=0\,.$$

Hence we may take

$$K=\operatorname{Re}\left[-\tfrac{1}{2}i\int_B^D\left(\frac{d\Omega}{dz}\right)^2 dz\right], \tag{1}$$

where now the integral should be taken along a path in $\mathfrak{D}$ connecting B and D.

3. *Three basic problems*

We consider three problems specified by the following boundary conditions for the magnetic potential ψ (cf. Fig. 3).

(i) $\psi=h$ on EFGHA,
$\psi=0$ on ABCDE.[2)]

1) This derivation is very simular to that of the Blasius formula in two-dimensional incompressible fluid dynamics.

2) Problem (i) has been treated by K.J.Binns [1] by essentially the same method as followed by us. Our numerical results agree with Binns's results (which have been published in graphical form only).

(ii) $\psi = h$ on GHABC,
$\psi = 0$ on CDEFG.

(iii) $\psi = h$ on ABC and EFG,
$\psi = 0$ on CDE and GHA.

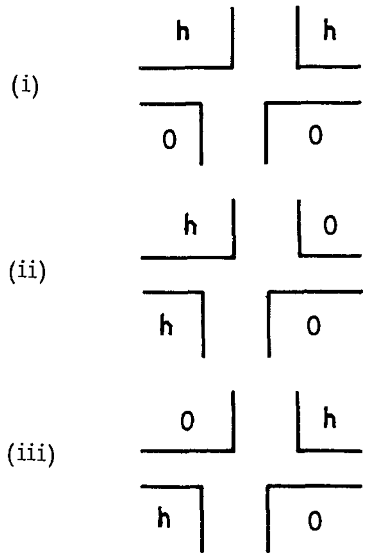

Fig. 3.

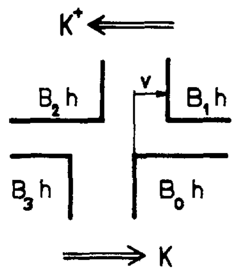

Fig. 4.

We denote the solutions of these three problems by ψ_1, ψ_2, ψ_3 and the corresponding complex potentials by Ω_1, Ω_2, Ω_3 (defined by $\mathrm{Im}\,\Omega_i(z)=\psi_i(x,y)$). If we have the general boundary conditions (cf. Fig. 4)

$$\psi = B_0 h \text{ on CDE}, \quad B_1 h \text{ on EFG}, \quad B_2 h \text{ on GHA}, \quad B_3 h \text{ on ABC},$$

where B_0, B_1, B_2, B_3 are arbitrary coefficients, then

$$\psi = A_1\psi_1 + A_2\psi_2 + A_3\psi_3 + B_0 h,$$

where

$$\begin{aligned} A_1 &= \tfrac{1}{2}(-B_0 + B_1 + B_2 - B_3), \\ A_2 &= \tfrac{1}{2}(-B_0 - B_1 + B_2 + B_3), \\ A_3 &= \tfrac{1}{2}(-B_0 + B_1 - B_2 + B_3). \end{aligned} \tag{2}$$

For the complex potential $\Omega(z)$, defined by $\mathrm{Im}\,\Omega(z) = \psi(x,y)$, we may take

$$\Omega(z) = A_1\,\Omega_1 + A_2\,\Omega_2 + A_3\,\Omega_3 + iB_0 h.$$

Substituting this into formula (1) we find for the horizontal force K on the lower slot

$$K = \sum_{m,n=1}^{3} A_m A_n K_{mn}, \tag{3}$$

where

$$K_{mn} = K_{nm} = \mathrm{Re}\left[-\tfrac{1}{2} i \int_B^D \left(\frac{d\Omega_m}{dz}\right)\left(\frac{d\Omega_n}{dz}\right) dz\right]. \tag{4}$$

It should be noted that K_{11}, K_{22}, K_{33} are the horizontal forces on the lower slot for the three basic problems.

For a fixed value of h, the coefficients K_{mn} are functions of v (the horizontal displacement of the upper slot relative to the lower slot). The parity of these functions can be found as follows.

Let us, for the moment, introduce the notation

$$K = K(B_o, B_1, B_2, B_3, v).$$

Reflecting Fig. 4 in a vertical line it is easily seen that

$$K(B_o, B_1, B_2, B_3, v) = -K(B_3, B_2, B_1, B_o, -v).$$

Together with (2) and (3) this implies

$$\sum_{m,n=1}^{3} A_m A_n K_{mn}(v) = -\sum_{m,n=1}^{3} A'_m A'_n K_{mn}(-v),$$

where $A'_1 = A_1$, $A'_2 = -A_2$, $A'_3 = -A_3$.

Since this has to be true for all A_1, A_2, A_3, it follows that

K_{11}, K_{22}, K_{33}, K_{23} are odd functions of v,
K_{12}, K_{13} are even functions of v.

Let $K^+ = K^+(B_o, B_1, B_2, B_3, v)$ be the horizontal force on the upper slot (in the direction as indicated in Fig. 4).

K^+ is connected with K as follows. By (1), we have

$$K - K^+ = \mathrm{Re}\left[-\tfrac{1}{2} i \left(\int_B^D + \int_F^H\right)\left(\frac{d\Omega}{dz}\right)^2 dz\right]$$

$$= \mathrm{Re}\left[\tfrac{1}{2} i \left(\int_D^F + \int_H^B\right)\left(\frac{d\Omega}{dz}\right)^2 dz\right], \tag{5}$$

since $\oint (d\Omega/dz)^2 dz = 0$ for the contour shown in Fig. 5.

Since

$$\frac{d\Omega}{dz} = \frac{\partial\psi}{\partial y} + i\,\frac{\partial\psi}{\partial x},$$

the integrand in (5) is real on the horizontal parts of the paths DF and HB. For $z \to \infty$ we have $d\Omega/dz \to B_1 - B_o$, and for $z \to -\infty$ we have $d\Omega/dz \to B_2 - B_3$.

Hence from (5) it follows that

$$K - K^+ = -\tfrac{1}{2} h (B_1 - B_o)^2 + \tfrac{1}{2} h (B_2 - B_3)^2. \tag{6}$$

Or, on substituting (2),

$$K - K^+ = -2h\, A_1 A_3. \tag{7}$$

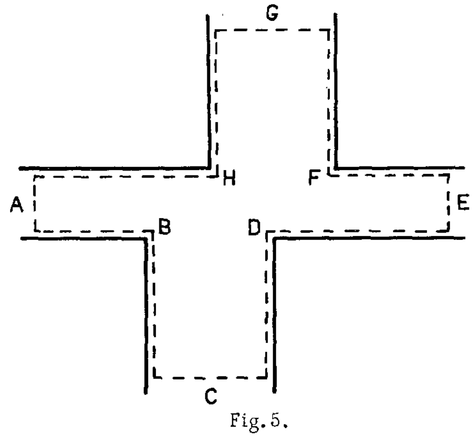

Fig. 5.

On the other hand, rotating Fig. 4 over 180 degrees, we see that

$$K^{+}(B_o, B_1, B_2, B_3, v) = K(B_2, B_3, B_o, B_1, v),$$

or, using (2) and (3),

$$K - K^{+} = 4A_1 A_3 K_{13} + 4A_2 A_3 K_{23}\,.$$

Comparing this with (7), we infer

$$K_{13} = -h/2, \quad K_{23} = 0. \tag{8}$$

Remark. It is possible to derive the relation (6) between K and K^{+} from the following formulation of the principle of virtual displacements: If, in an arrangement of conductors, each of which being kept at a *fixed potential,* one of the conductors is displaced by an infinitesimal distance, then the mechanical work done by the field on this conductor is equal to the *gain* in the total energy of the field. 1)

In order to apply this principle, we suppose the gap to be closed at the far left and at the far right by conducting pistons with, say, potentials zero. If we now move the upper and the lower slot simultaneously towards the right and keep the pistons fixed in space, then the rate of increase of the field energy is equal to the right-hand side of (6).

4. *Conformal mapping of the z-domain*

Coordinates in the z-plane are chosen as shown in Fig. 6. We want to map the domain $\mathfrak{D}$ conformally on the strip $0 < \mathrm{Im}\, t < \pi$ such that ABCDE is mapped on $\mathrm{Im}\, t = 0$, EFGHA on $\mathrm{Im}\, t = \pi$ and $z = 0$ on $t = \pi i/2$.

From the symmetry of $\mathfrak{D}$ it follows that the mapping function $t = f(z)$ satisfies $f(-z) = \pi i - f(z)$.

Consequently, if B, C and D are mapped on $t = \beta$, γ and δ, respectively, then F, G and H are mapped on $\pi i - \beta$, $\pi i - \gamma$ and $\pi i - \delta$, respectively. The transformation $w = e^{t}$ maps the strip $0 < \mathrm{Im}\, t < \pi$ on the upper half plane $\mathrm{Im}\, w > 0$.

With the Schwarz-Christoffel formula we find

$$\frac{dz}{dw} = C\,\frac{\left[(w-e^{\beta})(w-e^{\delta})(w+e^{-\beta})(w+e^{-\delta})\right]^{1/2}}{w(w-e^{\gamma})(w+e^{-\gamma})},$$

whence

$$\frac{dz}{dt} = C\,\frac{\left[(\sin t - \sinh\beta)(\sinh t - \sinh\delta)\right]^{1/2}}{\sinh t - \sinh\gamma}. \tag{9}$$

The parameters C, β, γ, δ have to be determined by the following relations:

(i) $\displaystyle\int_D^F \frac{dz}{dt}\, dt = v + ih.$

Taking the path of integration as shown in Fig. 7 we infer that

1) cf. [2], § 5.

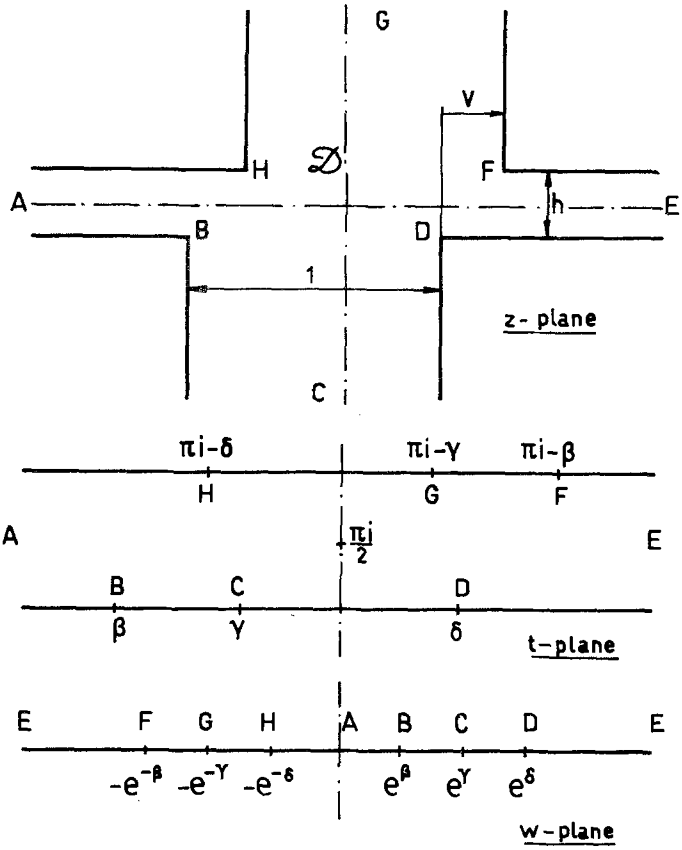

Fig. 6.

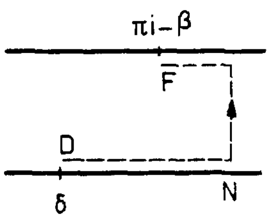

Fig. 7.

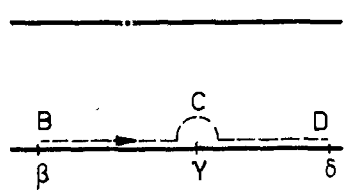

Fig. 8.

$$\pi C = h, \tag{10}$$

$$C \lim_{N\to\infty} \left(\int_{\delta}^{N} - \int_{-N}^{\beta} \right) \frac{|(\sinh t - \sinh \beta)(\sinh t - \sinh \delta)|^{1/2}}{|\sinh t - \sinh \gamma|} dt = v. \tag{11}$$

(ii) $\int_B^D \frac{dz}{dt} dt = 1.$

Taking the path of integration as shown in Fig. 8 we find

$$\pi C \frac{[(\sinh \gamma - \sinh \beta)(\sinh \delta - \sinh \gamma)]^{1/2}}{\cosh \gamma} = 1, \tag{12}$$

$$\fint_{\beta}^{\delta} \frac{[(\sinh t - \sinh \beta)(\sinh \delta - \sinh t)]^{1/2}}{\sinh t - \sinh \gamma} dt = 0, \tag{13}$$

where $⨍$ indicates that the Cauchy principal value has to be taken at the singularity $t = \gamma$.

5. *The potentials for the three basic problems*

We want to find the complex potentials Ω_1, Ω_2, Ω_3 for the three basic problems formulated in sec. 3.

It is obvious that for problem (i)

$$\Omega_1 = \frac{h}{\pi} t.$$

For problem (ii) and (iii) Ω is most easily found if we consider the corresponding boundary-value problems in the upper half of the w-plane, as indicated in Fig. 9.

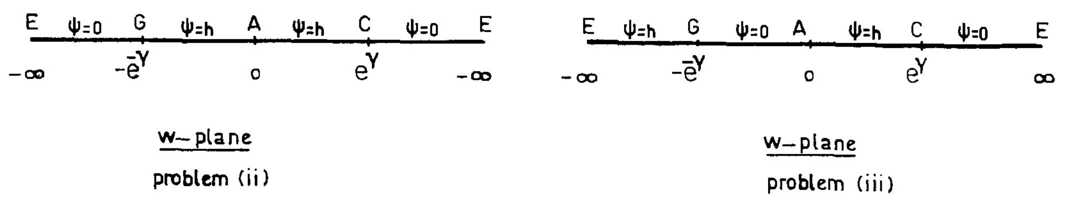

Fig. 9.

It is easily seen that the following functions satisfy the boundary conditions: for problem (ii)

$$\Omega_2 = \frac{h}{\pi} \log \frac{w - e^{\gamma}}{w + e^{-\gamma}} = \frac{h}{\pi} \log \frac{e^t - e^{\gamma}}{e^t + e^{-\gamma}},$$

for problem (iii)

$$\Omega_3 = \frac{h}{\pi} \log \frac{(w - e^{\gamma})(w + e^{-\gamma})}{w} = \frac{h}{\pi} \log(2 \sinh t - 2 \sinh \gamma).$$

For use in our formula for the magnetic force we list the derivatives of Ω_i:

$$\frac{d\Omega_1}{dt} = \frac{h}{\pi},$$

$$\frac{d\Omega_2}{dt} = \frac{h}{\pi} \frac{\cosh \gamma}{\sinh t - \sinh \gamma}, \tag{14}$$

$$\frac{d\Omega_3}{dt} = \frac{h}{\pi} \frac{\cosh t}{\sinh t - \sinh \gamma}.$$

6. *The magnetic force*

Substituting the results (9) and (14) into formula (4) we find the coefficients K_{mn} of the expression (3) for K.

$$K_{11} = -\frac{h}{2\pi}\int_\beta^\delta (\sinh t - \sinh\gamma)(\sinh t - \sinh\beta)^{-\frac{1}{2}}(\sinh\delta - \sinh t)^{-\frac{1}{2}}\,dt,$$

$$K_{22} = -\frac{h}{2\pi}\fint_\beta^\delta \cosh^2\gamma(\sinh t - \sinh\gamma)^{-1}(\sinh t - \sinh\beta)^{-\frac{1}{2}}(\sinh\delta - \sinh t)^{-\frac{1}{2}}\,dt,$$

$$K_{33} = -\frac{h}{2\pi}\fint_\beta^\delta \cosh^2 t(\sinh t - \sinh\gamma)^{-1}(\sinh t - \sinh\beta)^{-\frac{1}{2}}(\sinh\delta - \sinh t)^{-\frac{1}{2}}\,dt,$$

$$K_{12} = K_{21} = -\frac{h}{2\pi}\int_\beta^\delta \cosh\gamma(\sinh t - \sinh\beta)^{-\frac{1}{2}}(\sinh\delta - \sinh t)^{-\frac{1}{2}}\,dt,$$

$$K_{13} = K_{31} = -\frac{h}{2\pi}\int_\beta^\delta \cosh t(\sinh t - \sinh\beta)^{-\frac{1}{2}}(\sinh\delta - \sinh t)^{-\frac{1}{2}}\,dt,$$

$$K_{23} = K_{32} = -\frac{h}{2\pi}\fint_\beta^\delta \cosh\gamma\cosh t(\sinh t - \sinh\gamma)^{-1}(\sinh t - \sinh\beta)^{-\frac{1}{2}}(\sinh\delta - \sinh t)^{-\frac{1}{2}}\,dt.$$

According to (1) the integral from β to δ should have been taken along a path inside the strip $0 < \mathrm{Im}\ t < \pi$. However, deforming this path into a path as shown in Fig. 8 we get the above results since the semicircle around γ does not (in the limit) contribute.

The integrals for K_{13} and K_{23} can easily been calculated by the substitution

$$\sin\varphi = \frac{2\sinh t - \sinh\delta - \sinh\beta}{\sinh\delta - \sinh\beta}, \quad \sin\varphi_0 = \frac{2\sinh\gamma - \sinh\delta - \sinh\beta}{\sinh\delta - \sinh\beta}.$$

We find

$$K_{13} = -\frac{h}{2\pi}\int_{-\pi/2}^{\pi/2} d\varphi = -\frac{h}{2},$$

$$K_{23} = -\frac{h\cosh\gamma}{\pi(\sinh\delta - \sinh\beta)}\fint_{-\pi/2}^{\pi/2}\frac{d\varphi}{\sin\varphi - \sin\varphi_0} = 0.$$

This is in accordance with the results of sec. 3.

7. *The limiting case for* $|v| \to \infty$

In problem (i) of sec. 3 we obviously have $K_{11} \to 0$ for $|v| \to \infty$. For K_{22} of problem (ii), K_{33} of problem (iii) and K_{12} in formula (3) it will be shown that

$$K_{22} \to -(h/2)\mathrm{sgn}(v),\ K_{33} \to (h/2)\mathrm{sgn}(v),\ K_{12} \to -h/2 \ \text{for}\ |v| \to \infty.$$

For problem (ii) if $v \to -\infty$ and for problem (iii) if $v \to +\infty$, we have essentially the situation as shown in Fig. 10. For the mapping of the domain $\mathfrak{D}$ on the upper half plane $\mathrm{Im}\ w > 0$ in the manner as shown in Fig. 11,

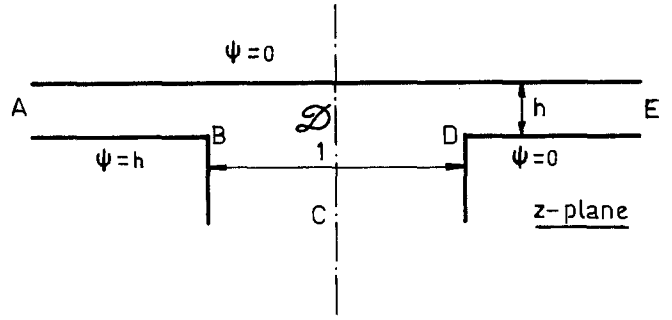

Fig. 10.

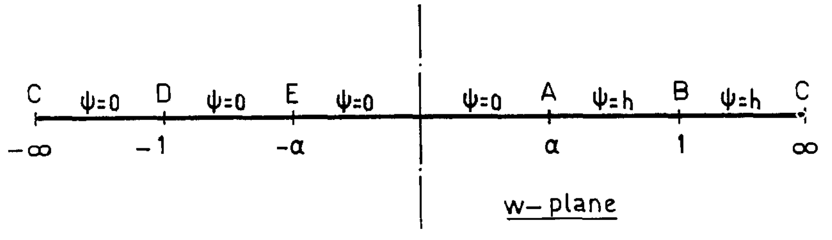

Fig. 11.

we find

$$\frac{dz}{dw} = \frac{1}{\pi} \frac{\sqrt{1-w^2}}{w^2-\alpha^2}, \qquad \frac{\sqrt{1-\alpha^2}}{2\alpha} = h.$$

It is readily seen that the complex potential may be taken to be

$$\Omega = -\frac{h}{\pi}\log(\alpha - w), \text{ whence } \frac{d\Omega}{dw} = \frac{h}{\pi(\alpha - w)}.$$

The use of formula (1) gives for the horizontal force on the lower slot

$$\lim_{v\to-\infty} K_{22} = \lim_{v\to\infty} K_{33} = \mathrm{Re}\left[\frac{h^2}{2\pi i}\int_B^D \frac{w+\alpha}{w-\alpha}\frac{dw}{\sqrt{1-w^2}}\right] = h/2.$$

The integral can be evaluated by suitable deformation of the contour, using the fact that the integrand is real for $-1 < w < 1$.

In order to find $\lim K_{12}$, we consider the situation $A_1=-1$, $A_2=1$, $A_3=0$ (cf. sec. 3).

If $v \to \infty$, this situation again essentially reduces to the situation of Fig. 10. Hence, (using (3))

$$\lim_{v\to\infty} (K_{11} + K_{22} - 2K_{12}) = h/2.$$

Since $K_{11} \to 0$, $K_{22} \to -h/2$, we find $K_{12} \to -h/2$.

The general results now follow from the fact that K_{22} and K_{33} are odd, and K_{12} is even in v.

8. *The reduction to standard elliptic integrals*

To compute the various integrals it is useful to reduce these to standard elliptic integrals. By the substitution

$$\tau = \frac{2\sinh t - \sinh\delta - \sinh\beta}{\sinh\delta - \sinh\beta}$$

and after considerable algebraic manipulation[1)] we can express the relation (13) and the integrals for K_{11}, K_{22}, K_{33}, K_{12} in terms of certain standard integrals which, with the help of [4], can be reduced to complete elliptic integrals of first and third kinds. The integral (11) for v can be handled by the substitution $\tau' = 1/\tau$.

In order to present the results we shall use the following abbreviations:

$$a = \tfrac{1}{2}(\sinh\delta - \sinh\beta),\quad b = \tfrac{1}{2}(\sinh\delta + \sinh\beta),$$

$$c = \tfrac{1}{2}(\sinh\delta + \sinh\beta - 2\sinh\gamma),\quad d = \frac{\cosh\delta + \cosh\beta}{\cosh\delta - \cosh\beta},$$

$$r = (\cosh\delta\,.\,\cosh\beta)^{-1/2},$$

$$p_1(\tau) = (1-\tau^2)(1+(a\tau+b)^2),$$

$$p_2(\tau) = (1+b^2)^{-1}(1-\tau^2)\,[a^2 + ((1+b^2)\tau + ab)^2],$$

$$\tau_1 = -c/a,\quad \tau_2 = -a/c,$$

$$I_{1j} = \int_{-1}^{+1} \frac{d\tau}{\sqrt{p_j}},\quad I_{2j} = \int_{-1}^{+1} \frac{\tau d\tau}{\sqrt{p_j}},\quad I_{3j} = \int_{-1}^{+1} \frac{d\tau}{(\tau-\tau_j)\sqrt{p_j}},\quad j = 1, 2.$$

We then get the relations corresponding to (7), (8) and (9), and expressions for the K_{mn}, as follows:

$$v = -\frac{h}{\pi c}\left(a^2 I_{12} - ac\, I_{22} - a\frac{\cosh^2\gamma}{h^2} I_{32}\right), \tag{15}$$

$$h^2(\sinh\delta - \sinh\gamma)(\sinh\gamma - \sinh\beta) = \cosh^2\gamma, \tag{16}$$

$$ac\, I_{11} - a^2 I_{21} + \frac{\cosh^2\gamma}{h^2} I_{31} = 0, \tag{17}$$

$$K_{11} = -\frac{h}{2\pi}(c\, I_{11} + a\, I_{21}), \tag{18}$$

$$K_{22} = -\frac{h}{2\pi}\frac{\cosh^2\gamma}{a} I_{31}, \tag{19}$$

$$K_{33} = -\frac{h}{2\pi}\left((c + 2\sinh\gamma)I_{11} + a\, I_{21} + \frac{\cosh^2\gamma}{a} I_{31}\right), \tag{20}$$

$$K_{12} = -\frac{h}{2\pi}\cosh\gamma\, I_{11}. \tag{21}$$

We would remark that by elimination of I_{11}, I_{21} and I_{31} from (18) to (21)

---

1) We are indebted to Mr.J.K.M.Jansen and Mr.H.Willemsen for performing this work.

it follows that

$$K_{11} + K_{22} - K_{33} + 2(\tanh\gamma)K_{12} = 0.$$

With the help of [3], Teil II, p. 47, formulae 1b, 2c and Teil I, p. 88 formula 8b. 12, we find

$$I_{11} = 2rK(k),$$

$$I_{21} = 2rd\left[\Pi(\rho_1, k) - K(k)\right],$$

$$I_{12} = 2rK(k'),$$

$$I_{22} = -2rd^{-1}\left[\Pi(\rho_1', k') - K(k')\right],$$

where

$$k^2 = \tfrac{1}{2} r^2(\cosh(\delta + \beta) - 1), \quad k'^2 = 1 - k^2 = \tfrac{1}{2} r^2(\cosh(\delta - \beta) + 1),$$

$$\rho_1 = \tfrac{1}{4} r^2(\cosh\delta - \cosh\beta)^2, \quad \rho_1' = -1 - \rho_1 = -\tfrac{1}{4} r^2(\cosh\delta + \cosh\beta)^2,$$

$$I_{31} = 2ar\left[\frac{h^2}{\cosh^2\gamma} C_1\Pi(\rho_2, k) - C_2K(k)\right],$$

$$I_{32} = 2r\left[\frac{h^2}{\cosh^2\gamma} C_3\Pi(\rho_2', k') - C_4K(k')\right],$$

with

$$C_1 = \frac{a(a - cd)}{ad - c}, \quad C_2 = \frac{1}{ad - c},$$

$$C_3 = -\frac{c(a - cd)}{ad - c}, \quad C_4 = -\frac{1}{a - cd},$$

$$\rho_2 = -\frac{h^2}{\cosh^2\gamma}(ad - c)^2\rho_1.$$

9. *Numerical computation*

For a given value of h we wish to find K as a function of v. It is simpler, however, to compute K and v as functions of the auxiliary variable γ. Then (16) and (17) are two equations with the unknowns β and δ from which β and δ can be computed. After that the computations of K_{11}, K_{22}, K_{33}, K_{12} and v are straightforward. For the computation of the elliptic integrals, we used the procedure from [4], [5].

The computation was carried out in FORTRAN on the IBM 1620 of the Technological University Eindhoven.

The graphs in Figs 12, 13, 14 and 15 have been produced by an on-line CALCOMP plotter, using plotting routines that are an implementation of ALGOL plotting procedures described in [6]. 1)

1) We are indebted to Mr. J. K. M. Jansen for the programming and for the careful preparation of the graphs.

10. Results

For various values of the parameter h (cf. Fig. 1) we have tabulated K_{11}, K_{22}, K_{33}, K_{12} as functions of v with an absolute error of less than 0.5×10^{-4}. For some values of h we have also given graphs of these functions (Figs 12 to 15).

These tables may be used for the general situation of Fig. 4 as follows. For given values of h and v the horizontal force K on the lower slot is

$$K = A_1^2 K_{11} + A_2^2 K_{22} + A_3^2 K_{33} + 2A_1 A_2 K_{12} - A_1 A_3 h,$$

where the coefficients A_1, A_2, A_3 have to be calculated according to formula (2) of sec. 3. For positive values of v the coefficients K_{11}, K_{22}, K_{33}, K_{12} may be taken from the tables.

For negative values of v one should utilise the fact that K_{11}, K_{22}, K_{33} are odd functions of v and K_{12} is an even function of v.

Finally, we note that the horizontal force K^+ on the upper slot is

$$K^+ = K + 2A_1 A_3 h.$$

h	v	K_{11}	K_{22}	K_{33}	K_{12}
	.0	.0000	.0000	.0000	-.0117
	.1	.0221	-.0019	.0278	-.0110
	.2	.0306	-.0036	.0395	-.0104
	.3	.0337	-.0052	.0445	-.0104
	.4	.0348	-.0070	.0471	-.0110
.1	.5	.0347	-.0091	.0485	-.0122
	.6	.0336	-.0118	.0493	-.0141
	.7	.0312	-.0156	.0497	-.0172
	.8	.0265	-.0215	.0499	-.0225
	.9	.0175	-.0315	.0500	-.0320
	1.0	.0047	-.0451	.0500	-.0452
	1.1	.0003	-.0497	.0500	-.0497
	.0	.0000	.0000	.0000	-.0234
	.1	.0228	-.0038	.0332	-.0228
	.2	.0351	-.0074	.0526	-.0221
	.3	.0407	-.0108	.0627	-.0222
	.4	.0429	-.0145	.0682	-.0234
	.5	.0429	-.0187	.0713	-.0256
.15	.6	.0411	-.0240	.0731	-.0291
	.7	.0371	-.0308	.0741	-.0345
	.8	.0301	-.0404	.0747	-.0427
	.9	.0190	-.0537	.0749	-.0549
	1.0	.0068	-.0675	.0750	-.0678
	1.1	.0012	-.0737	.0750	-.0737
	1.2	.0002	-.0748	.0750	-.0748
	1.3	.0000	-.0750	.0750	-.0750

Magnetic force on two staggered slotted half-planes

h	v	K_{11}	K_{22}	K_{33}	K_{12}
.2	.0	.0000	.0000	.0000	-.0379
	.1	.0218	-.0062	.0372	-.0374
	.2	.0360	-.0120	.0630	-.0369
	.3	.0434	-.0177	.0784	-.0374
	.4	.0466	-.0238	.0876	-.0391
	.5	.0469	-.0305	.0931	-.0424
	.6	.0446	-.0385	.0963	-.0474
	.7	.0396	-.0484	.0982	-.0547
	.8	.0314	-.0610	.0992	-.0650
	.9	.0200	-.0760	.0997	-.0781
	1.0	.0087	-.0898	.0999	-.0906
	1.1	.0025	-.0971	.1000	-.0973
	1.2	.0006	-.0994	.1000	-.0994
	1.3	.0001	-.0999	.1000	-.0999
	1.4	.0000	-.1000	.1000	-.1000
.25	.0	.0000	.0000	.0000	-.0546
	.1	.0201	-.0088	.0404	-.0543
	.2	.0348	-.0172	.0715	-.0542
	.3	.0434	-.0256	.0922	-.0551
	.4	.0474	-.0342	.1054	-.0575
	.5	.0479	-.0437	.1136	-.0617
	.6	.0455	-.0544	.1187	-.0680
	.7	.0400	-.0671	.1218	-.0768
	.8	.0314	-.0819	.1235	-.0882
	.9	.0205	-.0979	.1244	-.1014
	1.0	.0102	-.1118	.1248	-.1134
	1.1	.0038	-.1201	.1249	-.1207
	1.2	.0012	-.1235	.1250	-.1236
	1.3	.0004	-.1245	.1250	-.1246
	1.4	.0001	-.1249	.1250	-.1249
.3	.0	.0000	.0000	.0000	-.0732
	.1	.0183	-.0115	.0433	-.0732
	.2	.0325	-.0228	.0788	-.0734
	.3	.0416	-.0340	.1043	-.0748
	.4	.0461	-.0454	.1216	-.0779
	.5	.0470	-.0576	.1329	-.0829
	.6	.0445	-.0711	.1402	-.0902
	.7	.0390	-.0862	.1447	-.0999
	.8	.0308	-.1027	.1473	-.1118
	.9	.0208	-.1195	.1488	-.1248
	1.0	.0115	-.1336	.1495	-.1363
	1.1	.0052	-.1427	.1498	-.1439
	1.2	.0020	-.1472	.1499	-.1476
	1.3	.0007	-.1490	.1500	-.1491
	1.4	.0003	-.1496	.1500	-.1497
	1.5	.0001	-.1499	.1500	-.1499

h	v	K_{11}	K_{22}	K_{33}	K_{12}
	.0	.0000	.0000	.0000	-.1148
	.1	.0146	-.0173	.0484	-.1151
	.2	.0269	-.0345	.0909	-.1161
	.3	.0356	-.0515	.1247	-.1185
	.4	.0405	-.0687	.1498	-.1227
	.5	.0418	-.0863	.1677	-.1289
	.6	.0399	-.1047	.1799	-.1373
	.7	.0352	-.1239	.1881	-.1478
.4	.8	.0284	-.1431	.1933	-.1596
	.9	.0205	-.1612	.1964	-.1718
	1.0	.0130	-.1762	.1982	-.1824
	1.1	.0073	-.1869	.1991	-.1902
	1.2	.0038	-.1934	.1996	-.1950
	1.3	.0018	-.1968	.1998	-.1976
	1.4	.0009	-.1985	.1999	-.1989
	1.5	.0004	-.1993	.2000	-.1995
	1.6	.0002	-.1997	.2000	-.1998
	.0	.0000	.0000	.0000	-.1607
	.1	.0115	-.0231	.0526	-.1612
	.2	.0215	-.0461	.1007	-.1628
	.3	.0290	-.0689	.1412	-.1660
	.4	.0336	-.0916	.1733	-.1709
	.5	.0352	-.1143	.1976	-.1777
	.6	.0340	-.1371	.2153	-.1864
	.7	.0306	-.1597	.2278	-.1967
.5	.8	.0254	-.1813	.2362	-.2079
	.9	.0193	-.2009	.2418	-.2191
	1.0	.0135	-.2173	.2453	-.2289
	1.1	.0087	-.2296	.2473	-.2367
	1.2	.0052	-.2379	.2485	-.2421
	1.3	.0030	-.2432	.2492	-.2455
	1.4	.0017	-.2462	.2496	-.2475
	1.5	.0009	-.2479	.2498	-.2486
	1.6	.0005	-.2489	.2499	-.2493
	1.7	.0003	-.2494	.2499	-.2496
	.0	.0000	.0000	.0000	-.2094
	.1	.0090	-.0287	.0562	-.2100
	.2	.0169	-.0572	.1087	-.2121
	.3	.0231	-.0855	.1546	-.2156
	.4	.0271	-.1133	.1928	-.2209
	.5	.0288	-.1406	.2231	-.2278
	.6	.0283	-.1674	.2462	-.2364
	.7	.0259	-.1931	.2634	-.2461
	.8	.0222	-.2170	.2758	-.2565
.6	.9	.0177	-.2384	.2843	-.2667
	1.0	.0132	-.2564	.2901	-.2759
	1.1	.0093	-.2705	.2939	-.2834
	1.2	.0062	-.2808	.2962	-.2891
	1.3	.0040	-.2879	.2977	-.2931
	1.4	.0025	-.2926	.2986	-.2957
	1.5	.0015	-.2955	.2992	-.2974
	1.6	.0009	-.2973	.2995	-.2984
	1.7	.0005	-.2984	.2997	-.2991
	1.8	.0003	-.2990	.2998	-.2994

Magnetic force on two staggered slotted half-planes

h	v	K_{11}	K_{22}	K_{33}	K_{12}
	.0	.0000	.0000	.0000	-.2600
	.1	.0070	-.0339	.0593	-.2607
	.2	.0132	-.0675	.1154	-.2629
	.3	.0182	-.1007	.1657	-.2666
	.4	.0216	-.1332	.2090	-.2719
	.5	.0233	-.1648	.2447	-.2787
	.6	.0232	-.1951	.2731	-.2868
	.7	.0217	-.2238	.2951	-.2959
	.8	.0191	-.2502	.3116	-.3054
.7	.9	.0159	-.2736	.3236	-.3147
	1.0	.0125	-.2935	.3322	-.3232
	1.1	.0094	-.3096	.3381	-.3304
	1.2	.0067	-.3219	.3422	-.3362
	1.3	.0047	-.3309	.3449	-.3406
	1.4	.0031	-.3373	.3467	-.3437
	1.5	.0021	-.3417	.3479	-.3458
	1.6	.0014	-.3446	.3486	-.3473
	1.7	.0009	-.3465	.3491	-.3482
	1.8	.0006	-.3477	.3494	-.3489
	1.9	.0004	-.3486	.3496	-.3493
	.0	.0000	.0000	.0000	-.3118
	.1	.0054	-.0386	.0618	-.3126
	.2	.0103	-.0769	.1209	-.3148
	.3	.0144	-.1145	.1750	-.3184
	.4	.0172	-.1512	.2226	-.3235
	.5	.0187	-.1866	.2630	-.3300
	.6	.0190	-.2203	.2963	-.3375
	.7	.0181	-.2518	.3229	-.3459
	.8	.0163	-.2807	.3437	-.3545
	.9	.0140	-.3063	.3595	-.3631
.8	1.0	.0115	-.3283	.3712	-.3709
	1.1	.0090	-.3465	.3798	-.3778
	1.2	.0068	-.3610	.3860	-.3836
	1.3	.0050	-.3721	.3903	-.3881
	1.4	.0036	-.3803	.3934	-.3915
	1.5	.0026	-.3863	.3395	-.3941
	1.6	.0018	-.3906	.3969	-.3959
	1.7	.0012	-.3935	.3979	-.3972
	1.8	.0008	-.3956	.3986	-.3981
	1.9	.0006	-.3970	.3990	-.3987
	2.0	.0004	-.3980	.3994	-.3991

h	v	K_{11}	K_{22}	K_{33}	K_{12}
	.0	.0000	.0000	.0000	-.3644
	.1	.0042	-.0428	.0640	-.3651
	.2	.0081	-.0853	.1256	-.3672
	.3	.0113	-.1269	.1827	-.3707
	.4	.0137	-.1673	.2340	-.3755
	.5	.0151	-.2062	.2786	-.3815
	.6	.0155	-.2430	.3162	-.3885
	.7	.0150	-.2774	.3473	-.3961
	.8	.0139	-.3087	.3722	-.4040
	.9	.0122	-.3367	.3919	-.4117
.9	1.0	.0104	-.3609	.4017	-.4190
	1.1	.0085	-.3813	.4186	-.4256
	1.2	.0067	-.3980	.4272	-.4312
	1.3	.0052	-.4112	.4336	-.4358
	1.4	.0039	-.4215	.4383	-.4394
	1.5	.0029	-.4292	.4416	-.4422
	1.6	.0021	-.4350	.4440	-.4444
	1.7	.0015	-.4392	.4458	-.4459
	1.8	.0011	-.4423	.4470	-.4471
	1.9	.0008	-.4445	.4479	-.4479
	2.0	.0006	-.4461	.4485	-.4485
	2.1	.0004	-.4473	.4489	-.4490
	.0	.0000	.0000	.0000	-.4173
	.1	.0033	-.0466	.0658	-.4180
	.2	.0064	-.0927	.1295	-.4200
	.3	.0090	-.1379	.1892	-.4232
	.4	.0109	-.1817	.2436	-.4277
	.5	.0121	-.2237	.2918	-.4332
	.6	.0126	-.2635	.3334	-.4395
	.7	.0125	-.3005	.3685	-.4464
	.8	.0117	-.3343	.3975	-.4536
	.9	.0106	-.3647	.4210	-.4607
	1.0	.0092	-.3912	.4398	-.4675
1.0	1.1	.0078	-.4139	.4544	-.4736
	1.2	.0064	-.4329	.4658	-.4790
	1.3	.0051	-.4483	.4745	-.4836
	1.4	.0040	-.4606	.4811	-.4874
	1.5	.0031	-.4703	.4860	-.4904
	1.6	.0024	-.4778	.4897	-.4927
	1.7	.0018	-.4835	.4924	-.4946
	1.8	.0013	-.4878	.4944	-.4960
	1.9	.0010	-.4910	.4959	-.4970
	2.0	.0007	-.4933	.4970	-.4978
	2.1	.0005	-.4951	.4978	-.4984
	2.2	.0004	-.4964	.4984	-.4988

Magnetic force on two staggered slotted half-planes

h	v	K_{11}	K_{22}	K_{33}	K_{12}
	.0	.0000	.0000	.0000	-.9450
	.1	.0004	-.0664	.0742	-.9452
	.2	.0009	-.1322	.1475	-.9460
	.3	.0012	-.1971	.2194	-.9471
	.4	.0016	-.2606	.2890	-.9487
	.5	.0019	-.3222	.3558	-.9506
	.6	.0021	-.3815	.4193	-.9529
	.7	.0023	-.4383	.4791	-.9554
	.8	.0024	-.4922	.5349	-.9582
	.9	.0025	-.5429	.5865	-.9611
	1.0	.0025	-.5904	.6339	-.9640
	1.1	.0024	-.6345	.6771	-.9670
	1.2	.0023	-.6752	.7162	-.9699
	1.3	.0022	-.7124	.7514	-.9727
2.0	1.4	.0021	-.7463	.7828	-.9755
	1.5	.0020	-.7770	.8108	-.9780
	1.6	.0018	-.8045	.8355	-.9804
	1.7	.0016	-.8292	.8574	-.9827
	1.8	.0015	-.8511	.8765	-.9847
	1.9	.0013	-.8705	.8933	-.9866
	2.0	.0012	-.8877	.9079	-.9882
	2.1	.0010	-.9027	.9206	-.9897
	2.2	.0009	-.9159	.9317	-.9911
	2.3	.0008	-.9274	.9412	-.9922
	2.4	.0007	-.9374	.9495	-.9933
	2.5	.0006	-.9461	.9566	-.9942
	2.6	.0005	-.9536	.9628	-.9950
	2.7	.0005	-.9602	.9681	-.9957
	2.8	.0004	-.9658	.9727	-.9963
	2.9	.0003	-.9706	.9766	-.9968

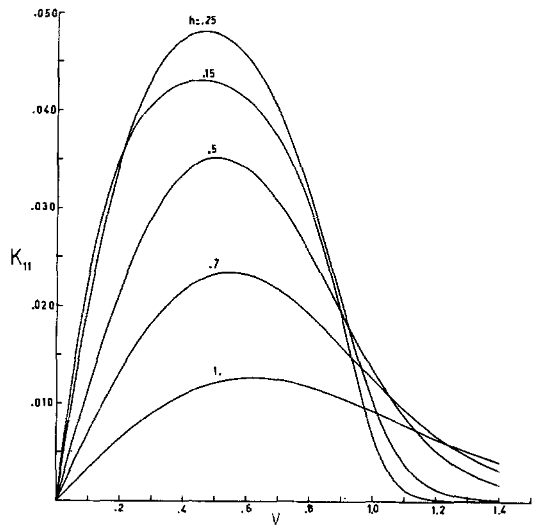

Fig. 12. K_{11} as function of v.

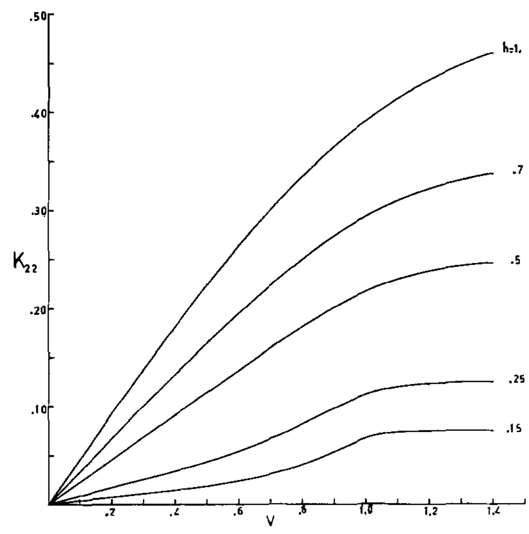

Fig. 13. K_{22} as function of v.

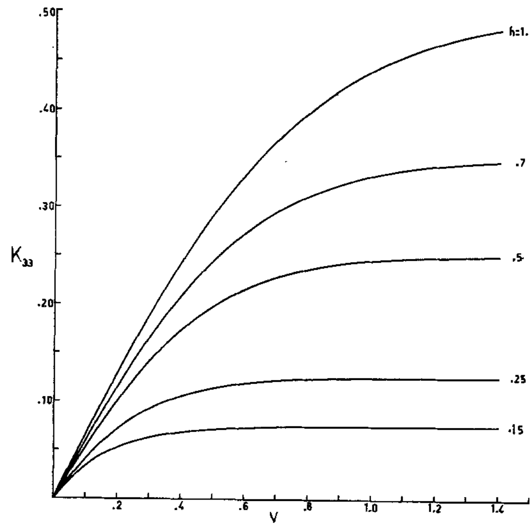

Fig. 14. K_{33} as function of v.

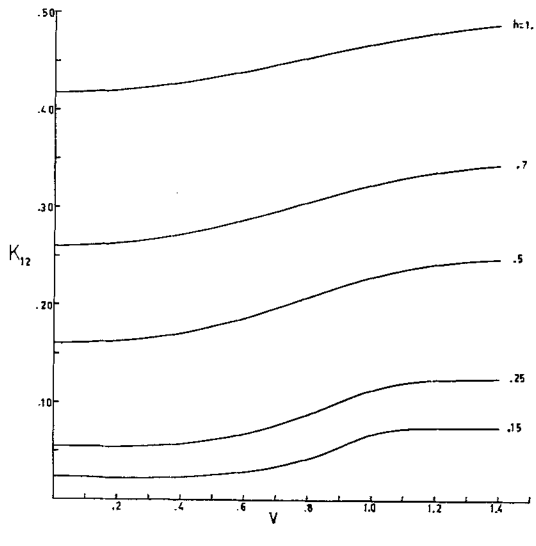

Fig. 15. K_{12} as function of v.

REFERENCES

1. K.J.Binns, The magnetic field and centring force of displaced ventilating ducts in machine cores. I.E.E.Proc., Part C, 108 (1961), 64-70.

2. L.D.Landau and E.M. Lifschitz, Electrodynamics of continuous media. Pergamon Press, Oxford, 1960.

3. W.Gröbner und N.Hofreiter, Integraltafel. Erster Teil, Unbestimmte Integrale, Dritte Auflage, Springer-Verlag, Wien 1961. Zweiter Teil, Bestimmte Integrale, Zweite Auflage, Springer-Verlag, Wien 1958.

4. R.Bulirsch, Numerical calculation of elliptic integrals and elliptic functions. Numer. Math., 7 (1965), 78-90.

5. id.id. 7 (1965), 353-354.

6. J.A.Th.M.van Berckel and B.J.Mailloux, Some ALGOL plotting procedures. Report MR 73, Mathematisch Centrum, Amsterdam, 1965.

[Received August 10, 1966]